Learning Unit 3: Numbers

The topic of numbers is essential in children's early mathematical development. Children encounter numbers early in everyday life—whether they are counting objects (such as blocks, steps, or apples), playing games (like dice games), following daily routines (for example 'Dinner will be ready in 5 minutes'), or listening to picture books that include numbers and quantities. In doing so, children already develop informal understandings of numbers, known as *number sense*.

Definition: Number Sense

The notion of number sense implies a conceptual understanding of number and number operations (Askew et al., 1997; McIntosh et al., 1997). Number sense is reflected by an awareness of the mathematical structure of numbers (Mulligan et al., 2020; Siemon et al., 2020) and by flexibility in using or creating various strategies for performing calculations (Greeno, 1991). To highlight this two-dimensional content of number sense, Wagner and Davis (2010) proposed that number sense is composed by (a) "quantity sense" which is a sense about amount and magnitudes and (b) "computation-based number sense" which is about manipulating numbers appropriately when combining them.

Development of Number Sense

Number sense develops over a long period of time and is reflected through a broad range of abilities. One of the foundational ideas that is indicative of the development of number sense in young children is the stable order of the *number sequence* (Case & Sandieson, 1991). Moreover, number sense is linked to understanding that there is one-to-one correspondence between a counted object in a set and the number word that it receives (Alibali & DiRusso, 1999). The principle of *cardinality* is also important; this refers to understanding that the last number word used to count a set of objects represents the total number of the objects in the

set (Geary, 1994) and presupposes anticipating that each counting number is quantitatively one more than the one before (Clements & Sarama, 2014). Furthermore, understanding the conservation of *quantity*, i.e., understanding the invariance of a set of objects when their arrangement changes, is an indicator of the development of number sense (Inhelder & Piaget, 1958).

Looking ahead, for older students, the development of number sense also includes the concept of place value. Students need to develop understanding of the base 10 numeration system. The central idea that they need to grasp is that the position of a digit in a number determines its value and that "10 of these is one of those' and '1000 of these is 1 of those' (Siemon et al., 2019).

Previous research has shown that these primary abilities with small numbers start to develop before children formally begin school and they form the basis for understanding operations with numbers and the structure of larger numbers (Jordan et al., 2007).

Number Sense: Two Groups of Foundational Tasks

While number sense is viewed by some researchers as an "intrinsic process" (Robinson et al., 2002), several tasks can be used in the classroom to support children to build on their intuitive number sense and refine it (Gersten & Chard, 1999). These tasks may vary in difficulty according to the age, level, and expertise of the children (Dyson et al., 2013).

In the following section two specific groups of foundational tasks are analyzed which have been considered as suitable for the development of number sense in children from Pre-K to Grade 1:

(a) Counting and enumeration tasks

(b) Number line tasks

(a) Counting and Enumeration

Counting and enumeration involve the production of the standard list of number words in order ("one, two, three, four, …") to count a set of a objects using one-to-one correspondence (Jordan et al., 2007). For small groups of numbers, children are expected to develop subitizing-based number recognition, i.e., rapidly recognizing the number of objects in the set without counting (Hannula et al., 2007). It is important to provide children with opportunities to count objects of varying sizes and quantities so that they can develop understanding of the counting procedure by being independent of these variables.

Difficulties and Strategies

Understanding the principle of cardinality is not a simple task for young children (Fosnot & Dolk, 2001). Children usually match counting words with objects, without essentially understanding that the last number word represents the amount of the set or the cardinality of the set (Fosnot & Dolk, 2001). For small numbers, children may "subitize" to obtain the correct number in the set, rather than using the cardinality principle; for this reason, they fail to answer how many when the numbers are larger and cannot easily be instantly recognized (Wynn, 1990). Moreover, some children have difficulties in counting sets from various starting points (Fuson, 1988).

One strategy that children usually apply to overcome errors in counting is counting loud; this helps them to reproduce the stable order of number correctly. Moreover, some children develop and use the "last word" rule, i.e., they understand that the last number word is the answer to "how many"; however, this strategy does not imply that they conceptually understand the principle of cardinality (Wynn, 1992). Moreover, children usually move an object to the side as it is counted to make sure that one-to-one correspondence is established, and an accurate answer is provided (Clements & Sarama, 2014).

Counting and enumeration skills can be cultivated and assessed through a variety of tasks. The following list provides a synthesis of indicative examples:

- Reproducing that stable order of numbers, forwards and backwards, from any given number.
- Reading and representing numbers using objects, pictures, number line, words, and symbols.
- Matching number words to objects in a one-to-one correspondence.
- Counting items to determine how many.
- Organizing items to keep track of what has been counted and what is yet to be counted.
- Instantly recognizing (subitizing) the number of objects in small sets, without counting.
- Recognizing common number arrangements, such as on a standard dice.
- Making correspondences between sets of objects.
- Expressing equality of groups, using phrases like “same as”.
- Using familiar spatial patterns and structures to support counting. For example, collecting items in twos and then counting by twos.
- Count in sequence forwards and backwards by twos, fives, and tens.
- Partitioning numbers in terms of their parts based on part-part-whole relationships.
- Identifying whole quantities as the result of recognizing smaller quantities based on part-part-whole relationships.

In addition to counting and enumeration skills, number line tasks are relevant, as mentioned below.

(b) Ordinal Understanding of Numbers/ Number Line

The number line is a “representation of numbers on a straight line where points represent numbers and the distance between points matches the arithmetical difference between the corresponding numbers” (Heeffer, 2011, p. 865). It represents the basic idea of the number series and is therefore used to develop an ordinal understanding of numbers (numbers used

for ordering) (Diezmann & Lowrie, 2007). Furthermore, the number line is essential for building a relational number concept, that is, for interpreting numbers in relation to other numbers (e.g., 28 is between 20 and 30, closer to 30).

It has been shown that students' performance on number line tasks is closely related to their overall mathematical achievement (Schneider et al., 2018).

Types of Number Lines

There are several types of number lines that differ in the information displayed about numbers. On a proportional number line, for example, only the starting and endpoints are marked and labelled with numbers and can be seen as "boundary points" (Teppo & van den Heuvel-Panhuizen, 2014, p. 48). Marked number lines (also called "structured number lines", e.g., Diezmann & Lowrie, 2007, or "filled number lines", e.g., Teppo & van den Heuvel-Panhuizen, 2014) have more hatch marks, ranging from fully marked with hatch marks and labelled with numbers (e.g., for whole numbers) to partially marked and labelled (e.g., Schulz & Wartha, 2021).

The number line offers particular opportunities, as it can be used across several school levels, has a high degree of abstraction, and allows flexibility in terms of types of numbers (e.g., whole numbers or integers), number ranges (e.g., 0 10 or 0 100) and types of tasks (Schulz & Wartha, 2021; Teppo & van den Heuvel-Panhuizen, 2014). The large number of changeable elements on the number line means that the number line can be used in many ways.

Difficulty level of number line tasks can be varied, for example, by choosing different number ranges (Schulz & Wartha, 2021) and by choosing different numbers of displayed hatch marks (labelled with numbers).

In order to promote the ordinal understanding of numbers through the use of the number line, it is advisable to start by working on the fully marked number line. As the learning process progresses, fewer and fewer hatch marks and labelled numbers can be given. The aim is that students are able to confidently translate between numbers and the corresponding positions on a proportional number line and to use task-adequate strategies (Schulz & Wartha, 2021). The aim is not only for the children to correctly locate numbers on the number line, but also

for them to develop strategies for quickly finding positions or number words and number symbols and mastering them. This aspect will become clearer in the next section.

Difficulties and Strategies

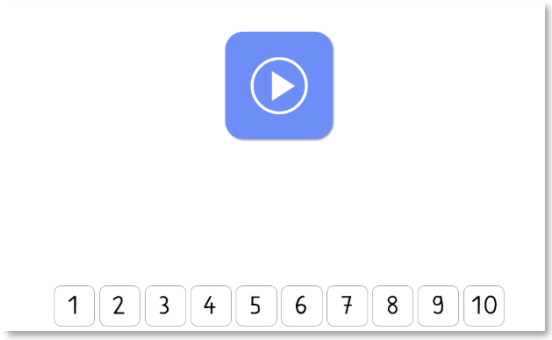
There are various strategies for children to use with the number line. Strategies for locating numbers on the number line may differ with respect to the use of the different reference points. It was shown that the use of these reference points (e.g., the midpoint of the number line) can positively influence number line performance (e.g., Peeters et al., 2016; 2017). However, as numbers have often to be interpreted in relation to each other to locate numbers on number lines (Schulz & Wartha, 2021), especially for students with mathematical difficulties, the number line is a complex matter and often difficult to understand. Previous research has shown that students with mathematical difficulties tend to have difficulties in number line tasks, such as lower accuracy in locating numbers and lower flexibility in using strategies (e.g., van't Noordende et al., 2016). Studies examining students' strategy use on marked number lines (Simon & Schindler, 2022; Simon et al., 2023) showed that students with mathematical difficulties used less advantageous strategies than students without mathematical difficulties and that they used strategies less task-adequate: Students with mathematical difficulties used counting strategies more often than students without mathematical difficulties, and they showed less use of nearest reference points than students without mathematical difficulties.

Important Skills Regarding Number Sense

Based on the topics presented, three key skills are highlighted below. If difficulties arise in these areas, they can be addressed with MADITA App which is presented in a separate learning unit. The three areas are as follows:

Number Recognition:

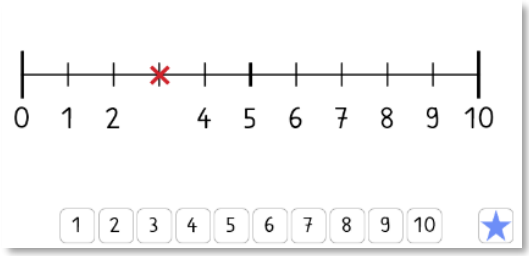
It is important that children can recognize the symbolic representation of numbers. To check this, you can, for example, say a number to the children and they must find the correct number from a preselected range (see example 1). So, they must identify the symbolic representation of numbers by choosing the number they hear. First graders can usually do this with numbers up to 10, and some children can even do it with numbers up to 20.



Example 1

Number Sequence:

It is also important that students can figure out the next number, starting from 1 or any number on the number line (see example 2). Usually, children can find the missing number at least up to 10, and in some cases up to 20.



Example 2

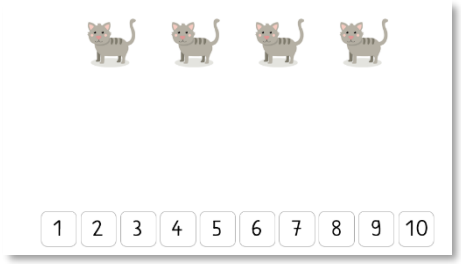
Counting:

In addition, children should be able to count when the number is represented by horizontally organized pictures/dots (see example 3). This is particularly important in early mathematical

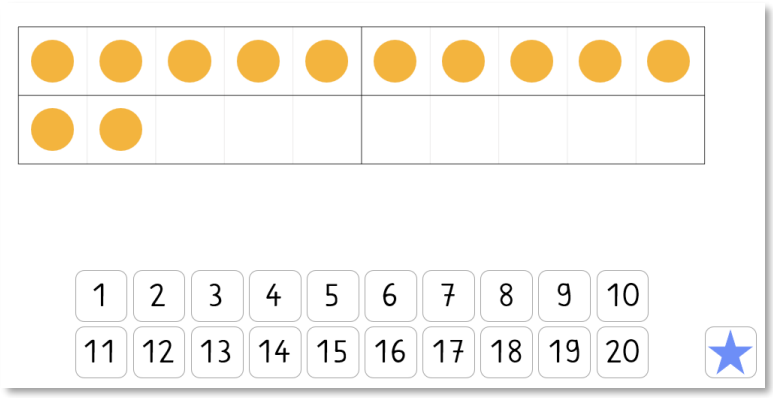
development when working with numbers up to 5, although some children can already do this up to 10.

It is also important to count correctly when the numbers are presented on ten or twenty frames (see example 4). This can also be practiced using flash vision, in which children must identify the patterns on the 20-frame after it briefly disappears.

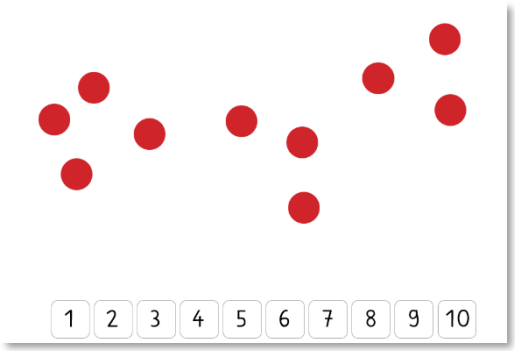
Even if the numbers are represented by pictures that are not organized, children in early mathematics should master this at least up to 5 or 10 (see example 5).



Example 3



Example 4



Example 5

The learning unit on numbers has shown how important this area is. How numbers are promoted in the MADITA app is explored in more depth in a separate learning unit.

The next learning unit relates to the topic of comparing.

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